Finite time computation of A-infinity algebra structures on Ext algebras

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Outline

Setting the scene

Let

- k be a field
- R a k-algebra
- (P_*, d) a minimal projective resolution of k over R

Computation of Ext-algebras

Recall that

- $\mathrm{Ext}_R(k,k)$ is chain homotopy classes of chain maps $P_* \to P_*$.
- $\mathrm{End}_R(P_*)$ is the graded R-module of graded maps $P_* \to P_*$.
- $\mathrm{End}_R(P_*)$ is a dg-algebra with $\partial f = df + (-1)^{|f|} fd$.
- Cycles in $\mathrm{End}_R(P_*)$ are graded chain maps.
- Boundaries are null-homotopic maps.

Hence, $\mathrm{Ext}_R(k,k) = H_* \mathrm{End}_R(P_*)$.

A-infinity algebras

Recall that

Definition

An A_∞-algebra A is a graded vector space equipped with a family of multilinear operations $\{m_k : A^{\otimes k} \to a\}_{k\geq 1}$ of degrees $|m_k| = k-2$ fulfilling the Stasheff axioms:

$$\mathrm{St}_k : \sum_i \pm m_{n-i+1} \circ_j m_i = 0$$

A-infinity algebra morphisms

Definition

A morphism $A \to B$ of A_∞-algebras is a family of multilinear maps $\{f_k : A^{\otimes k} \to B\}_{k \geq 1}$ of degrees $|f_k| = k - 1$ fulfilling the Stasheff morphism axioms

$$\mathrm{St}_k^m : \sum_i \pm f_{n-i+1} \circ_j m_i = \sum_{r, \sum i_k = n} \pm m_r \circ (f_{i_1} \otimes \cdots \otimes f_{i_r})$$

There are formulae for the signs in all these axioms. As the core of my arguments will end up showing that the signs are not perturbed, I will refrain from stating the signs explicitly.

Outline

Introduction

The Kadeishvili algorithm

Results

The Minimality theorem

Theorem (Kadeishvili 1980, many many others)

Suppose A is a dg-algebra. Then there is an A_∞-structure on H_*A such that $m_1 = 0$, m_2 is induced by the multiplication of A, and there is a quasi-isomorphism of A_∞-algebras $f_* : A \to H_*A$. Furthermore, if A has a unit 1, then we can choose the A_∞-structure $\{m_*\}$ on H_*A and the morphism f_* such that $m_k(a_1, \ldots, 1, \ldots, a_k) = 0$ and $f_k(a_1, \ldots, 1, \ldots, a_k) = 0$.

The Kadeishvili algorithm

Reading Kadeishvili's original proof yields an algorithm for the computation of parts of an A_∞-structure.
Input: elements $a_1, \ldots, a_k \in A$.
Output: function values $m_k(a_1, \ldots, a_k)$, $f_k(a_1, \ldots, a_k)$.

1. Compute

$$U_k(a_1, \ldots, a_k) = \sum_{1<i<k-1} \pm f_{k-i+1}(a_1, \ldots, m_i(a_s, \ldots, a_{s+i}), \ldots, a_k) - \sum_{r \neq 1,k} \pm m_r(f_{i_1}(a_1, \ldots, a_{i_1}), \ldots, f_{i_r}(a_{k-i_r}, \ldots, a_k))$$

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1. Compute $U_k(a_1, \ldots, a_k)$. This depends only on lower arity operations.
2. $U(a_1, \ldots, a_k)$ is a cycle, so it belongs to a homology class. Call this class $m_k(a_1, \ldots, a_k)$.

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3. $f_1(m_k(a_1, \dots, a_k))$ is homologous to $U(a_1, \dots, a_k)$, so there is some $w \in A$ such that $\partial w = U(a_1, \dots, a_k) - f_1(m_k(a_1, \dots, a_k))$. Set $f_k(a_1, \dots, a_k) = w$.
4. Return $m_k(a_1, \dots, a_k)$ and $f_k(a_1, \dots, a_k)$.

Drawbacks with the Kadeishvili algorithm

- Will only compute the value of the structure maps for one single parameter set.
- Unless R has finite projective dimension, $\mathrm{Ext}_R(k, k)$ has infinitely large basis.
- Kadeshvili's algorithm won't give a complete A_∞-structure on its own.

Outline

S-linearity

Definitions

- An A_n-algebra structure is an A_∞-structure, truncated at m_n: only $m_1, \ldots, m_n$ are defined and only $\mathrm{St}_1, \ldots, \mathrm{St}_n$ need to hold. Similarily, morphisms between A_n-algebras are defined.
- We call an A_n or A_∞ algebra A S-*linear*, for $S \subseteq A$ a commutative, associative subalgebra, if for any $a_1, \ldots, a_k \in A$, $s \in S$:

$$m_k(a_1, \ldots, sa_i, \ldots, a_k) = sm_k(a_1, \ldots, a_i, \ldots, a_k)$$

- Similarily, we will call a morphism $f_* : A \to B$ of A_∞-algebras S-linear if for any $a_1, \ldots, a_k \in A$ and $s \in S$ (m_2 denoted by $\cdot$):

$$f_k(a_1, \ldots, sa_i, \ldots, a_k) = f_1(s) \cdot f_k(a_1, \ldots, a_i, \ldots, a_k)$$

S-linearity to the rescue

Theorem (V-J)

For R, k, P_* as given in the first slide, write $A = \mathrm{End}_R(P_*)$ and suppose that

- *There is a polynomial subalgebra $S = k[z_1, \ldots, z_s] \subseteq H_*A$.*
- *H_*A is free as an S-module.*
- *We have an S-linear A_{n-1}-algebra structure on H_*A and an S-linear morphism of A_{n-1}-algebras $H_*A \to A$.*
- *All elements $f_1(s)$, for $s \in S$ commute with all elements on the form $f_k(a_1, \ldots, a_k)$ for $k < n$.*

Then we can choose m_n and f_n so that we get an S-linear A_n-algebra structure.

How does S-linearity help?

- If H_*A is a *finite* free S-module, then S-linearity helps us reduce the computational workload to compute an A_n-algebra structure with Kadeishvili's algorithm to a finite workload.
- Regardless of finiteness conditions, S-linearity allows us to excise parts of H_*A from consideration.
- We get a single recognition criteria while computing iteratively: as long as all $f_1(s)$ commute with all $f_k(a_1, \ldots, a_k)$, this method works.

Proof sketch

By investigating the potential terms of $U_k(a_1, \dots, a_k)$, we show that the S-linearity and commutativity conditions globally admit

$$U_k(a_1, \dots, sa_i, \dots, a_k) = f_1(s) U_k(a_1, \dots, a_i, \dots, a_k)$$

Signs remain fixed since S is a *commutative* subalgebra and therefore necessarily concentrated in even degrees.

Finite time computation

Theorem (V-J)

Suppose that A is a dg-algebra and that H_*A admits an A_{2q-2}-algebra structure with a corresponding quasiisomorphism $H_*A \to A$ such that all structure maps m_k and all quasiisomorphism component maps f_k vanish for $q \leq k \leq 2q-2$. Then this A_{2q-2}-algebra structure extends to an A_∞-algebra structure with no further non-trivial maps introduced.

Proof.

follows by showing that for all terms in the higher Stasheff morphism axioms, at least one component falls within the interval $q \leq k \leq 2q-2$, and hence vanishes. □

These two theorems combined enable us – for good enough algebras A – to compute A_∞-algebra structures inductively and in finite time.

Computations

- Let $R = k[x]/x^n$. The A_∞-algebra structure on $\mathrm{Ext}_R(k, k)$ may be computed in finite time with this approach. The resulting structure was first demonstrated by Madsen (2002).
- Suppose G is a finite p-group. Then k has a periodic resolution over kG if G is cyclic or generalized quaternionic. The cyclic case is the one above. Preliminary investigations indicate that the quaternoinic unit group has too rich structure to be amenable to this approach.
- For more complex groups – such as for instance the dihedral groups – an approach with regular sequences might work. The current state of these results, however, does not seem to help.

Conclusions

- The naïve approach to computation does work. In special cases, it can be made to work in finite time.
- Whether or not the finite time computation simplifications work can be discovered while computing.
- Whether or not the entire structure is already computed can be discovered while computing.
- The classes of algebras for which this has been made to work are embarrassingly few.
- More examples and more work needed.

Any questions?