Persistent Cohomology and Circle-valued coordinates

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Outline

1. Motivation: Intrinsic coordinates
2. Theory: Persistent cohomology and circle-valued maps
3. Practice: Finding and interpreting parametrizations

Coordinatization

Essentially

It's all about finding *coordinate function* on a dataset $X \subseteq \mathbb{R}^d$.
Preferably few coordinates - cognitive tools.

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- Principal Component Analysis

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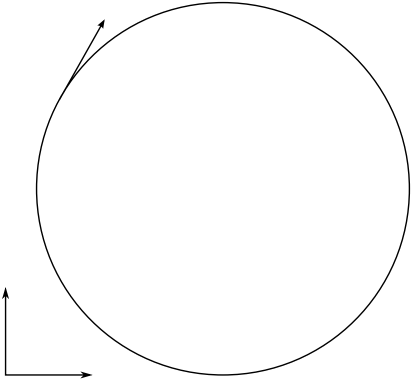
Recently

- Nonlinear methods: drop expectation that for f coordinate: $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$.
- Isomap, Kernel methods, Locally linear methods

Problem cases

In order to find few intrinsic coordinates, we want to stick close to the local dimension.



Some shapes take up too many coordinates.

Locally 1-dimensional. Globally 2 coordinates needed to describe all points. The shape doesn't fit in $\mathbb{R}$.

Similar problems arise with sphere and torus.

Suggested fix

Circle-valued coordinates

- Use $S^1 = [0,1]/(0 \sim 1)$ as additional coordinate space
- Fixes the circle
- Fixes the torus
- Occurs naturally:
  - Phase coordinates for waves
  - Angle coordinates for directions
  - Any recurrent phenomenon

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Circle-valued coordinates

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Circle-valued coordinates and cohomology

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Persistent cohomology

- Degree one cohomology equivalent to circle-valued maps
- Persistence picks out relevant features from noise
- Once a feature-rich parameter has been found, we can work in ordinary (non-persistent) cohomology theories

We compute persistent cohomology by using a special case of the zig zag persistence algorithm.
(coming soon to an auditorium near you!)

From cohomology to circle-valued parametrizations

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  Needed for the isomorphism: $H^1(X;\mathbb{Z})$.
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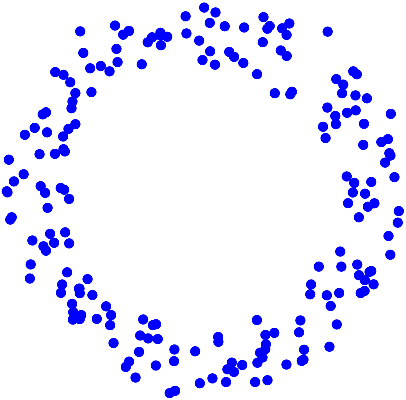
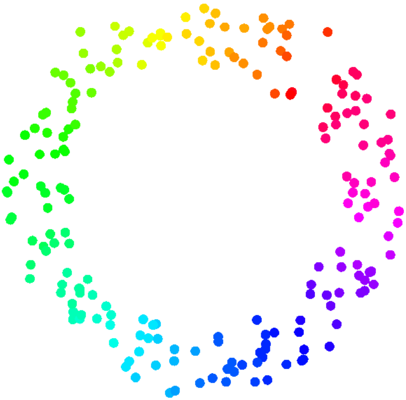
  If $H^2(X;\mathbb{Z})$ has no p-torsion, lift $H^1(X;\mathbb{Z}/p) \to H^1(X;\mathbb{Z})$.
- The representative chains for $H^1(X;\mathbb{Z})$ yields very non-smooth maps: all data points $\mapsto 0$. All edges wrap around the target circle.
  We can smooth a cocycle in $C^1(X;\mathbb{Z})$ to a harmonic cocycle in $C^1(X;\mathbb{R}) \cap C_1(X;\mathbb{R})$ belonging to the same cohomology class in $H^1(X;\mathbb{Z})$.

Outline

Parametrized circles



Parametrized circles



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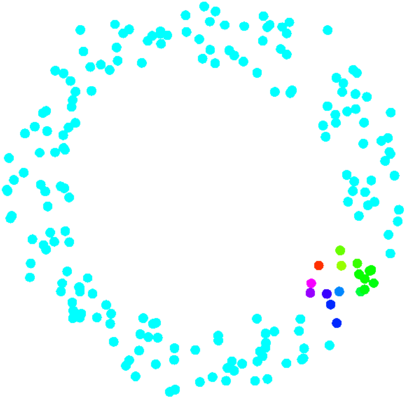

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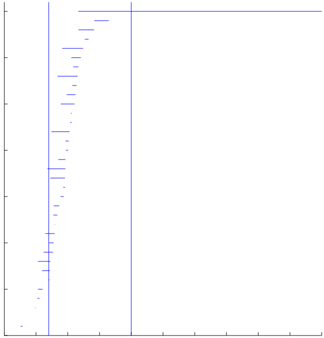
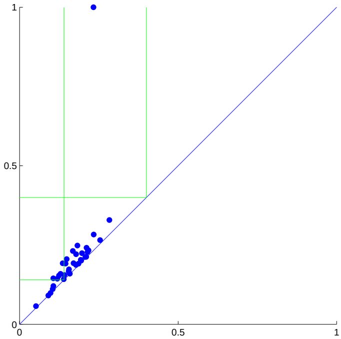
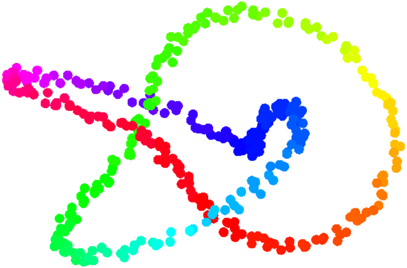


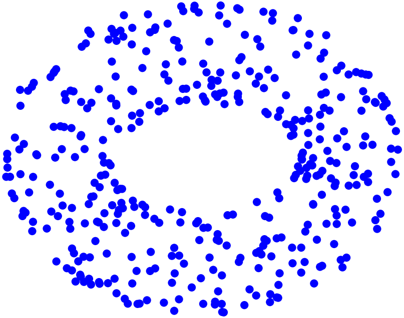


Knots and links



Knots and links

Torus



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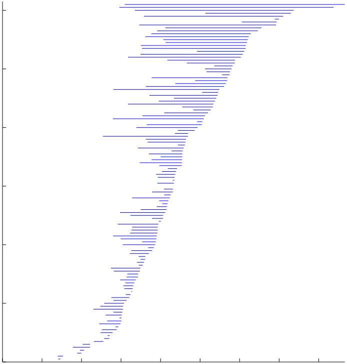

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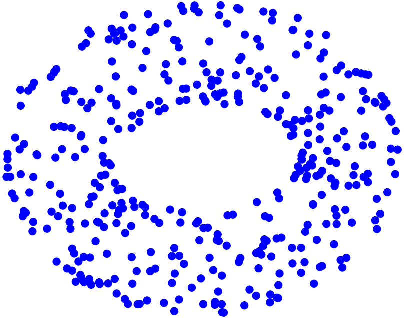

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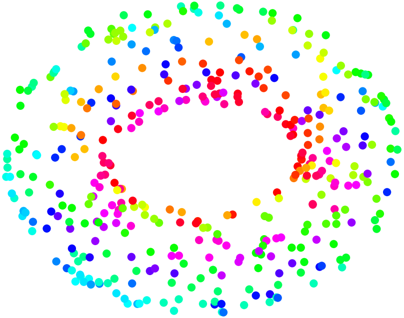

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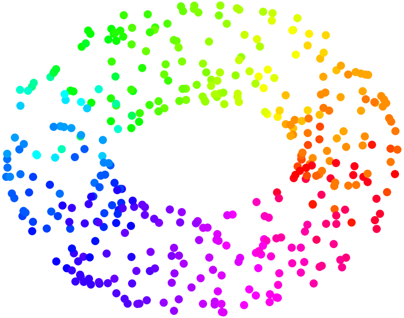

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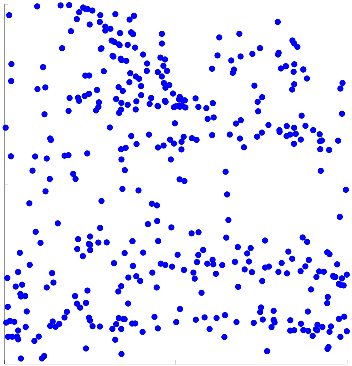
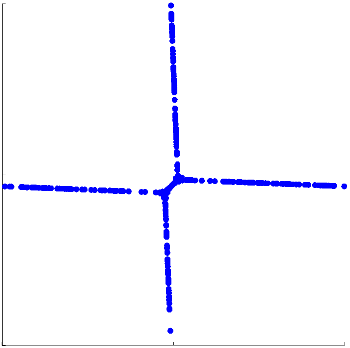


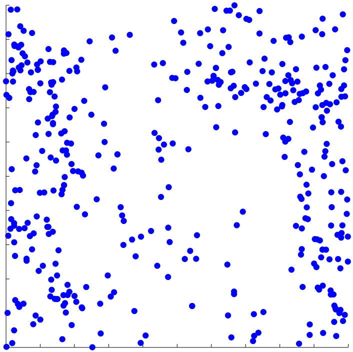
Correlation plot for this torus parametrization

Torus



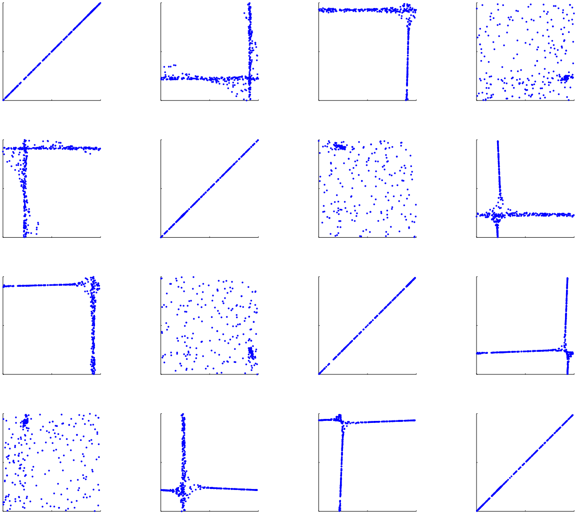
Correlation plot for a wedge of two circles

Torus

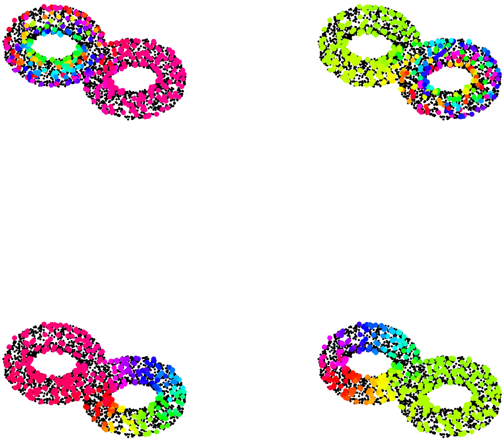


Correlation plot for an elliptic curve in $\mathbb{C}P^2$

Pop quiz



Pop quiz



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Questions?